Category $\mathcal{O}$: Methods

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1 Extensions in Category $\mathcal{O}$

Proposition 1

Let $\lambda, \mu \in \mathfrak{h}^*$. Then

(a) Let M be a highest weight module of weight μ with $\lambda \geq \mu$ or λ is not comparable to μ. Then $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M) = 0$. In particular

$$\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), L(\lambda)) = \mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M(\lambda)) = 0, \quad \mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), L(\mu)) = 0$$

(b) If $\lambda > \mu$ and $N(\lambda)$ is the maximal submodule of $M(\lambda)$, then

$$\mathrm{Hom}_{\mathcal{O}}(N(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(\mu))$$

(c) $\mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$

Proof. (a) Recall there is a bijection of sets $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), M)$ with equivalence classes of SES

$$0 \to M \to E \xrightarrow{\pi} M(\lambda) \to 0$$

where $\underline{E \in \mathcal{O}}$. It therefore suffices to show any such sequence splits. Let $v_\lambda \in M(\lambda)_\lambda$ be the h.w. vector. Let $\widetilde{v_\lambda}$ be any lift of v_λ under π; we claim $\widetilde{v_\lambda}$ is a h.w. vector of weight λ in E. Notice this will give us the required splitting as the universal property of $M(\lambda)$ will give us a map $\varphi : M(\lambda) \to E$ sending $v_\lambda \mapsto \widetilde{v_\lambda}$ and you can check that this map has to be injective (Use PBW basis in $M(\lambda)$). Because $\underline{E \in \mathcal{O}}$, it is $\mathfrak{h}-$semisimple, and thus we can write $\widetilde{v_\lambda}$ as a sum of weight vectors $\widetilde{v_\lambda} = \sum_{i=1}^{n} a_i v_i$ where $v_i \in E_{\gamma_i}$. Since π is a $\mathfrak{g}$ module morphism we have that $\pi(E_\gamma) \subseteq M(\lambda)_\gamma$ for $\gamma \in \mathfrak{h}^*$ and thus

$$M(\lambda)_\lambda \ni v_\lambda = \pi(\widetilde{v_\lambda}) = \sum_{i=1}^{n} a_i \pi(v_i) \in M(\lambda)_{\gamma_1} \oplus \ldots \oplus M(\lambda)_{\gamma_n}$$

But the weight space decomposition of $M(\lambda)$ is a direct sum decomposition and thus $\gamma_i = \lambda$ for all i. Thus we see that $\widetilde{v_\lambda} \in E_\lambda$.

Because $\pi(e_i \cdot \widetilde{v_\lambda}) = e_i \cdot \pi(\widetilde{v_\lambda}) = e_i \cdot v_\lambda = 0$ we have that $e_i \cdot \widetilde{v_\lambda} \in M$. However this means that $\lambda + \alpha_i$ is a weight of M, and since M is h.w. of weight μ, this means that

$$\lambda + \alpha_i = \mu - \sum k_j \alpha_j \implies \mu - \lambda = \sum k_j^* \alpha_j, \quad k_j^* \in \mathbb{Z}^{\geq 0}$$

where $k_i^* \in \mathbb{Z}^{\geq 1}$, aka $\lambda < \mu$ which is contrary to assumption. It follows that we have $e_i \cdot \widetilde{v_\lambda} = 0$ and thus $\widetilde{v_\lambda}$ is a h.w. vector of weight λ as desired.

(b) From the SES

$$0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$$

we get the LES in cohomology

$$\ldots \to \mathrm{Hom}_{\mathcal{O}}(M(\lambda), L(\mu)) \to \mathrm{Hom}_{\mathcal{O}}(N(\lambda), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}}(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), L(\mu)) \to \ldots$$

$\mathrm{Hom}_{\mathcal{O}}(M(\lambda), L(\mu)) = 0$ because $\lambda > \mu$ so there's nowhere for the h.w. vector in $M(\lambda)$ to go but zero as the image of a h.w. vector of weight λ under a $\mathfrak{g}-$module morphism is a h.w. vector of weight λ. Since $\lambda > \mu$, $\mathrm{Ext}^1_{\mathcal{O}}(M(\lambda), L(\mu)) = 0$ by part (a) so this completes the proof.
(c) Replace μ with λ above. The last term is still zero by part (a) so it suffices to show $\mathrm{Hom}_{\mathcal{O}}(N(\lambda), L(\lambda)) = 0$. Because $L(\lambda)$ is simple, any nonzero map $\phi : N(\lambda) \to L(\lambda)$ is surjective. But as $N(\lambda) \in \mathcal{O}$, the same argument in (a) shows that any lift of v_λ, say $\widetilde{v_\lambda} \in N(\lambda)_\lambda$. But by the construction of $N(\lambda)$, $N(\lambda)_\lambda = 0$ and thus ϕ can't be surjective and thus has to be the zero map.

Warning. Category $\mathcal{O}$ is not closed under extensions. In fact it's not even closed under extensions of Verma modules. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\lambda \in \mathfrak{h}^*$. Let N be the 2-dimensional $U(\mathfrak{b})$ module where

$$e \cdot v = 0 \ \forall v \in N, \quad h \leftrightarrow \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Let $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$. Then we then have an exact sequence

$$0 \to M(\lambda) \to M \xrightarrow{\pi} M(\lambda) \to 0$$

that is not split! This amounts to checking that any element in $\pi^{-1}(v_\lambda)$ is not a h.w. vector of weight λ. Thus $M \notin \mathcal{O}$ as otherwise this sequence would split.

2 Duality in Category $\mathcal{O}$

The natural choice for a duality functor is to send each module $M \mapsto M^*$ where the action of $U(\mathfrak{g})$ on M^* is given by

$$(x \cdot f)(m) := -f(x \cdot m)$$

coming from the antipode in the hopf algebra $U(\mathfrak{g})$. However recall that for M is an infinite-dimensional module, then M^* is an even larger infinite-dimensional module and thus has no chance to be in $\mathcal{O}$. However recall that weight spaces of M are f.d. Thus our first try for a duality functor will be

Definition 2.1 (Try 1)**.**

$$M^{*\vee} := \bigoplus_\lambda (M_\lambda)^*$$

where the action of $U(\mathfrak{g})$ on $(M_\lambda)^*$ is given above.

Now will $M^{*\vee} \in \mathcal{O}$? A quick computation shows that the answer is NO.

Example 1. Let $\mathfrak{g} = \mathfrak{sl}_2$ and consider $M = M(2)$. Let $v_k = f^k v_0/k! \in M_{2-2k}$ where v_0 is the h.w. vector.

[Draw usual picture of actions here]

Let $\varphi_k = v_k^*$ be the dual basis vector to v_k. Now compute that

$$(e \cdot \varphi_3)(v) = -\varphi_3(e \cdot v)$$

Since φ_3 is only nonzero on the weight space M_{-4}, it follows we must have $v \in M_{-6}$ in order for the number above to be nonzero. In particular we see that

$$(e \cdot \varphi_3)(v_4) = -\varphi_3(e \cdot v_4) = -\varphi_3((-1)v_3) = 1 \implies e \cdot \varphi_3 = \varphi_4$$

Iterating this process shows that $e^k \varphi_2 = k!\varphi_{2+k}$ and in particular is never 0 and thus not locally $U(\mathfrak{n})$ finite.

However the example above tells us what we should do to achieve locally $U(\mathfrak{n})$ finiteness; we should interchange the actions of e and f. Now because $(g \cdot h \cdot f)(v) = f(h \cdot g \cdot v)$, in order to obtain a new left $\mathfrak{g}-$module structure on V^* we need to precompose with a lie algebra anti-automorphism[1] of $\mathfrak{g}$ instead of a lie algebra automorphism. As such we define

Definition 2.2. *Consider the lie algebra anti-automorphism τ of $\mathfrak{g}$ given by sending $e_\alpha \mapsto f_\alpha$, $f_\alpha \mapsto e_\alpha$ and fixing h_α[2]. Then define the twisted action of $\mathfrak{g}$ on M^* by*

$$(x \cdot_\tau f)(m) := f(\tau(x) \cdot m)$$

From now on we will just write $x \cdot f = x \cdot_\tau f$.

Definition 2.3. *Let M be a $U(\mathfrak{g})$ module which is $\mathfrak{h}$ semisimple with f.d. weight spaces. Then the (BGG) dual of M is defined as a set by*

$$M^\vee = \bigoplus_{\lambda \in \mathfrak{h}^*} (M_\lambda)^*$$

where the $\mathfrak{g}$ module structure on $(M_\lambda)^*$ is given by the twisted τ action above.

Lemma 2.4. *Let M satisfy the conditions above. Then*

(1) $\left(M^\vee\right)_\lambda = (M_\lambda)^*$

(2) $\operatorname{ch}(M^\vee) = \operatorname{ch}(M)$

(3) $L(\lambda)^\vee = L(\lambda)$

Proof. (1) Because $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, we always have an exact sequence of $\mathbb{C}$ vector spaces

$$0 \to \bigoplus_{\lambda \neq \mu \in \mathfrak{h}^*} M_\mu \to M \to M_\lambda \to 0$$

So dualizing as vector spaces gives

$$0 \to (M_\lambda)^* \to M^* \to \left(\bigoplus_{\lambda \neq \mu \in \mathfrak{h}^*} M_\mu \right)^* \to 0$$

This means that we can identify

$$(M_\lambda)^* = \left\{ f \in M^* \mid f|_{M_\mu} = 0 \ \forall \mu \neq \lambda \right\}$$

[1] $\varphi([x,y]) = [\varphi(y), \varphi(x)] = -[\varphi(x), \varphi(y)]$

[2] For matrix lie algebras this is just taking the transpose.

Now given $f \in (M_\lambda)^*$ since it vanishes outside of M_λ, it's completely determined by what it does on M_λ. But for $v \in M_\lambda$ we have

$$(h \cdot f)(v) := f(\tau(h) \cdot v) = f(h \cdot v) = f(\lambda(h)v) = \lambda(h)f(v) \quad \forall h \in \mathfrak{h}$$

In other words $(M_\lambda)^* \subseteq (M^\vee)_\lambda$. Conversely, if $f \in (M^\vee)_\lambda$ and $f|_{M_\mu} \neq 0$ the same calculation above shows that on M_μ, $h \cdot f = \mu(h)f$ contrary to assumption and thus we have $(M^\vee)_\lambda \subseteq (M_\lambda)^*$.

(2) is a direct consequence of (1) as $\dim V = \dim V^*$ for V f.d.
(3) is a direct consequence of (2) as f.d. modules of $\mathfrak{g}$ are completely determined by their characters. Edit: only works for λ integral dominant. Instead use that $^\vee$ is an exact contravariant functor (see below) so that if M is not simple, then $M^\vee$ is not simple, as

$$0 \to A \to M \to M/A \to 0 \implies 0 \to (M/A)^\vee \to M^\vee \to A^\vee \to 0$$

It follows that since $(L(\lambda)^\vee)^\vee = L(\lambda)$ is simple, so is $L(\lambda)^\vee$. One then checks that v_λ^* is a h.w. vector in $L(\lambda)^\vee$ of weight λ where v_λ is a h.w. vector in $L(\lambda)$ and since there is a unique simple module of h.w. λ we have $L(\lambda)^\vee \cong L(\lambda)$.

Theorem 2
The BGG dual $^\vee$ *satisfies*

(a) $^\vee$ *is an exact (contravariant) functor on the category of* $\mathfrak{g}$ *modules which are* $\mathfrak{h}$ *semisimple with f.d. weight spaces.*

(b) $^\vee$ *descends to a functor* $^\vee : \mathcal{O} \to \mathcal{O}$ *such that* $M \mapsto M^{\vee\vee}$ *is isomorphic to the identity functor.*

(c) For any $M \in \mathcal{O}$ *and any central character* χ, $(M^\vee)^\chi \cong (M^\chi)^\vee$. *In particular* $^\vee$ *descends to a functor* $^\vee : \mathcal{O}_\chi \to \mathcal{O}_\chi$.

(d) Let $M, N \in \mathcal{O}$. *Then* $(M \oplus N)^\vee = M^\vee \oplus N^\vee$. *Thus* M *indecomposable* $\implies$ $M^\vee$ *indecomposable.*

(e) $\mathrm{Ext}^1_{\mathcal{O}}(M, N) = \mathrm{Ext}^1_{\mathcal{O}}(N^\vee, M^\vee)$.

Proof. (a) As soon as you check $\vee$ is a functor you are done because in the category of vector spaces $M_\lambda \mapsto M_\lambda^*$ is exact.
(b) Let $\phi \in M_\lambda^\vee$. Note that $e_\alpha \cdot \phi \in M_{\lambda+\alpha}^*$. This is because for any $\mathfrak{g}$ module N, we have $e_\alpha \cdot N_\lambda \subseteq N_{\lambda+\alpha}$. Apply this to $N = M^\vee$ and by Lemma 2.4 we have that $(M^\vee)_{\lambda+\alpha} = M_{\lambda+\alpha}^*$.

Now suppose $M \in \mathcal{O}$. $M^\vee$ will then have a weight decomposition given by dualizing each M_λ by Lemma 2.4 and so is $\mathfrak{h}$ semisimple. Because the set of weights of M is contained the union of the cones $\cup_i^n \mu_i - \Gamma_i$ [Draw picture], it follows that $M_{\mu+\sum k_i\alpha_i} = 0$ for $k_i >> 0 \implies M^\vee_{\mu+\sum k_i\alpha_i} = 0$ for $k_i >> 0$. By above we showed that each e_α takes us to a higher weight space and thus $U(\mathfrak{n}) \cdot v = 0$ for any v. Finally to show that $M^\vee$ is f.g as a $U(\mathfrak{g})$ module it suffices to show it has finite length as recall that given

$$0 \to A \to B \to C \to 0$$

A, C f.g. $\implies B$ f.g. and the "base case" $L(\mu)$ is clearly f.g. as a $U(\mathfrak{g})$ module. Consider the last step of a composition series for M

$$0 \to N \to M \to L(\mu) \to 0$$

Applying the functor $^\vee$ yields

$$0 \to L(\mu)^\vee \to M^\vee \to N^\vee \to 0$$

But by Lemma 2.4 $L(\mu)^\vee = L(\mu)$ and more importantly simple so this gives us the start of a composition series for $M^\vee$. Now repeat the same procedure but with N. Because $\mathcal{O}$ has finite length, we see this procedure eventually stops and we will have produced a finite composition series for $M^\vee$ as desired. (In fact given a fixed composition series for M, this process gives us a composition series for $M^\vee$ whose factors are the same as our fixed composition series for M, but the order of the factors are reversed!).
$(c) - (e)$ Left to the reader.

2.1 Duals of H.W. modules

Example 2 (Dual Verma modules in $\mathfrak{sl}_2$)**.** Consider our previous example with $M(2)$. Recall that our twisted action was specifically designed so that e still raises weights by 2 instead of lowering by 2. Thus in terms of arrows, our picture of $M(2)^\vee$ is exactly the same as $M(2)$.

[Draw $M(2)$ here] [Draw $M(2)^\vee$ here]

However the scalars by which e, f move v_i change. Essentially to get e to move say v_{-3}^* up, we need to use the actual action of f on v_{-2} and therefore the scalar by which we move up by is exactly 3 and in general we have [picture]. Using this perspective one can see why $L(2)$ is self dual; when we cut off $M(2)$ by the maximal submodule, the scalars by which we move up by in $L(2)^\vee$ is exactly the same as in $L(2)$ but in reverse order. This symmetry allows us to define an isomorphism (it essentially says we only need to check relations for one of F or E) $L(2)^\vee \xrightarrow{\sim} L(2)$ given by $v_2^* \mapsto 2v_2, v_1^* \mapsto v_1, v_0^* \mapsto 2v_0$[3].

Theorem 3
Let $\lambda, \mu \in \mathfrak{h}^*$. Then

(a) The dual Verma module $\nabla(\lambda) := M(\lambda)^\vee$ has $L(\lambda)$ as its unique simple submodule and its other composition factors $L(\mu)$ satisfy $\mu < \lambda$.

(b) If M is a h.w. module of weight λ, then $M^\vee$ is also a highest weight module of weight λ.

(c) Any nonzero homomorphism $M(\lambda) \to M(\lambda)^\vee$ has the simple submodule $L(\lambda)$ as its image. Moreover we have

$$\dim \operatorname{Hom}_{\mathcal{O}}(M(\lambda), M(\lambda)^\vee) = 1, \quad \dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = 0 \ \text{ for } \mu \neq \lambda$$

(d) $\operatorname{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)^\vee) = 0$ for all λ, μ.

Proof. (a) is just the dual of the fact that a h.w. module has a unique maximal submodule and thus unique maximal quotient. The second statement follows from the fact that the composition factors of $M^\vee$ is the same as of M as proved in previous theorem.

[3]In general for $L(n)$ the formula will be $\Phi(v_0^*) = nv_0$ and $\Phi(v_k^*) = \dfrac{k!}{(n-1)\dots(n-k+1)} v_k$ for $k \geq 1$.

(b) Left to the reader. (c) The image I, of a nonzero morphism $M(\mu) \to M(\lambda)^\vee$ is a nonzero submodule of $M(\lambda)^\vee$ of h.w. μ. I therefore contains a simple submodule but by (a) it follows that $L(\lambda) \hookrightarrow I$ and thus $\lambda \leq \mu$. On the other hand, as I is a h.w. module of weight μ it has $L(\mu)$ as a quotient and since $I \hookrightarrow M(\lambda)^\vee$ this shows that $L(\mu)$ appears as a composition factor of $M(\lambda)^\mu$ and so again by (a) we have that $\mu \leq \lambda \implies \mu = \lambda$. In this case, we see that

$$M(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow M(\lambda)^\vee \tag{1}$$

gives us a nonzero morphism, and since $\dim M(\lambda)^\vee_\lambda = \dim M(\lambda)_\lambda = 1$, it follows that up to scalar this is the only nonzero morphism.
(d) If $\lambda \geq \mu$ or incomparable, then since $M(\mu)^\vee$ is a h.w. module of weight μ by (b), by Proposition 1 we have that $\operatorname{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)^\vee) = 0$. Otherwise we must have $\mu \geq \lambda$ but in this case for any SES

$$0 \to M(\mu)^\vee \to M \to M(\lambda) \to 0$$

corresponding to an element in $\operatorname{Ext}^1_{\mathcal{O}}(M(\lambda), M(\mu)^\vee)$, we can dualize to obtain

$$0 \to M(\lambda)^\vee \to M^\vee \to M(\mu) \to 0$$

which corresponds to an element of $\operatorname{Ext}^1_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = 0$ as $\mu \geq \lambda$. Dualize again and use that BGG dual commutes with direct sums.

Remark. Eq. (1) exhibits $L(\lambda)$ as an "intermediate extension" in geometry. Moreover parts (b) and (c) can be thought of geometrically as the statement that $j^*_\lambda (j_\mu)_! = 0$ for $\lambda \neq \mu$.

3 Standard Filtrations

Definition 3.1. *An object $M \in \mathcal{O}$ has a standard filtration if there is a sequence of submodules*

$$0 \subset M_1 \subset \ldots \subset M_n = M$$

s.t. M_i/M_{i-1} is isomorphic to a Verma module. Denote by $(M : M(\lambda))$ the number of times $M(\lambda)$ appears

Remark. Note that because $\mathcal{O}$ is of finite length, any standard filtration must be finite or otherwise you can construct a composition series of infinite length.

Proposition 3.2. *Suppose $M \in \mathcal{O}$ has a standard filtration. Then*

(a) Suppose λ is a maximal weight[4] *then M has a submodule isomorphic to $M(\lambda)$ and $M/M(\lambda)$ has a standard filtration.*

(b) If $M = M_1 \oplus M_2$ then M_1, M_2 also have standard filtrations.

(c) M is free as a $U(\mathfrak{n}^-)$ module. (Analogue of PBW basis in $M(\lambda)$)

[4] This means that $\nexists \mu \in P(M)$ s.t. $\mu > \lambda$. This will imply that all vectors in M_λ are h.w. vectors, but the converse isn't true. For $\mathfrak{g} = \mathfrak{sl}_2$, consider $M = M(2) \oplus M(0)$. Then notice M_0 is two dimensional and contains a h.w. vector. However, since $M_2 \neq 0$, 0 is not a maximal weight. On the other hand 2 is a maximal weight. Also note that λ being a maximal weight is also not the same as M_λ consisting of only highest weight vectors, consider $M = L(1) \oplus M(-3)$, we have that M_{-3} only has h.w. vectors but -3 is not maximal.

Proof. (a) Let m_λ be any vector in M_λ. Because $e_\alpha \cdot m_\lambda \in M_{\lambda+\alpha} = 0$ since λ is a maximal weight of M, it follows that m_λ is a h.w. vector of weight λ. By the universal property of $M(\lambda)$, we have a map $\varphi : M(\lambda) \to M$. We claim it's injective. Because M has a standard filtration, let i be the smallest index for which $\varphi(M(\lambda)) \subset M_i$. Thus we see that the reduction map

$$\overline{\varphi} : M(\lambda) \xrightarrow{\varphi} M_i \xrightarrow{\pi} M_i/M_{i-1}$$

is nonzero. But by definition $M_i/M_{i-1} \cong M(\mu)$ for some μ. Thus $\lambda \le \mu$ or otherwise there's nowhere for the h.w. vector of $M(\lambda)$ to go. But because λ is a maximal weight of M and thus of M_i, we must actually have $\lambda = \mu$. [Draw picture] Since any nonzero endomorphism of $M(\lambda)$ must send the h.w. vector to the h.w. vector $\overline{\varphi}$ must be an isomorphism and thus φ is injective.

Notice that $M_{i-1} \cap M(\lambda) = 0$ as given $x \in M_{i-1} \cap M(\lambda)$, we see that $\pi \circ \varphi(x) = 0 \implies x = 0$ as $\overline{\varphi}$ is an isomorphism. As a result $M_{i-1} \to M/M(\lambda)$ is injective (as the kernel is $M_{i-1} \cap M(\lambda)$) and by the Third isomorphism Theorem we have the exact sequence

$$0 \to M_{i-1} \to M/M(\lambda) \to M/M_i \to 0$$

The side factors have standard filtrations and thus they combine to give a standard filtration for $M/M(\lambda)$.

(b) Sketch: Use induction on standard filtration length and wlog one can find $M(\lambda)$ inside M by part (a) s.t.

$$M/M(\lambda) = M_1 \big/ M(\lambda) \oplus M_2$$

Part (a) tells us that $M/M(\lambda)$ has a standard filtration and so by induction we conclude that M_2 has a standard filtration and $M_1 \big/ M(\lambda)$ has a standard filtration ($\implies$ M_1 has a standard filtration).

(c). Proceed by induction on standard filtration length. The base case is true because $M(\lambda)$ has basis $F_1^{e_1} \dots F_k^{e_k} \otimes 1$ by PBW which of course is a $U(\mathfrak{n}^-)$ basis. For the induction step by part (a) we can find a submodule $M(\lambda) \hookrightarrow M$

$$0 \to M(\lambda) \to M \to M \big/ M(\lambda) \to 0$$

s.t. $M \big/ M(\lambda)$ has a standard filtration. By induction it follows that $M \big/ M(\lambda)$ is a free $U(\mathfrak{n}^-)$ module. But since free $U(\mathfrak{n}^-)$ modules are projective, it follows that the above sequence splits as $U(\mathfrak{n}^-)$ modules and thus we have

$$M \overset{U(\mathfrak{n}^-)\ mod}{=\!=\!=} M(\lambda) \oplus \frac{M}{M(\lambda)}$$

But both summands on the RHS are free $U(\mathfrak{n}^-)$ modules and thus so is M.

Remark. Part (b) above is actually deeper than you think. Given a standard filtration $\{M_i\}_{i\in I}$ for M and $N \subset M$ a submodule, $\{N \cap M_i\}_{i\in I}$ is not necessarily a standard filtration for N as

$$\frac{M_i \cap N}{M_{i-1} \cap N} \cong \text{ submodule of } \frac{M_i}{M_{i-1}} = M(\mu)$$

by the second isomorphism theorem. Thus if <u>all submodules</u> of Verma modules are also Verma modules, then $\{N \cap M_i\}_{i\in I}$ would give us a standard filtration of N. But this isn't true once you are in rank 2 or higher, e.g. $\mathfrak{sl}_3$. So <u>arbitrary</u> submodules of modules with a standard filtration need not have a standard filtration.

Theorem 4
Suppose M has a standard filtration, then for all $\lambda \in \mathfrak{h}^*$, we have

$$(M : M(\lambda)) = \dim \operatorname{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee)$$

Proof. You guessed it, we will proceed by induction on the standard filtration length of M. For the base case we clearly have that $(M(\mu) : M(\lambda)) = \delta_{\lambda\mu}$ while by Theorem 3 we have that $\dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = \delta_{\lambda\mu}$ so they agree. For the induction step, since M has a standard filtration, we have an exact sequence $0 \to N \to M \to M(\mu) \to 0$ where N also has a standard filtration, for some $\mu \in \mathfrak{h}^*$ and thus a LES in cohomology

$$0 \to \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) \to \operatorname{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee) \to \operatorname{Hom}_{\mathcal{O}}(N, M(\lambda)^\vee) \to \operatorname{Ext}^1_{\mathcal{O}}(M(\mu), M(\lambda)^\vee)$$

The last term is zero by Theorem 3 and since N has a standard filtration by definition, we see that by induction

$$\begin{aligned}\dim \operatorname{Hom}_{\mathcal{O}}(M, M(\lambda)^\vee) &= \dim \operatorname{Hom}_{\mathcal{O}}(N, M(\lambda)^\vee) + \dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) \\ &= (N : M(\lambda)) + \delta_{\lambda\mu}\end{aligned}$$

But the last term above is literally $(M : M(\lambda))$ by the exact sequence $0 \to N \to M \to M(\mu) \to 0$ and this completes the induction step.

4 Refined Composition Factors of $M(\lambda)$

Question 1
For a fixed weight λ, what are conditions on μ s.t. $L(\mu)$ appears a composition factor of $M(\lambda)$?

We can give some necessary conditions to the question above. Namely, suppose we have a composition series of $M(\lambda)$ where $M_k/M_{k-1} \cong L(\mu)$. As central characters descend to submodules and quotients, it follows that $\chi_\lambda = \chi_\mu$ and by Harish-Chandra this means that $\boxed{\mu = w \star \lambda}$. Moreover since $M_k \in \mathcal{O}$, we know by the same trick as in Proposition 1 that a lift of the h.w. vector $v_\mu \in L(\mu)$ must be in $(M_k)_\mu \subseteq M(\lambda)_\mu$ and thus μ is a weight of $M(\lambda)$. But this means that $\boxed{\mu \leq \lambda}$.

However it turns out we can give a more refined condition. For each $\lambda \in \mathfrak{h}^*$ we will define a subgroup $W_{[\lambda]}$ of the Weyl group as follows.

Definition 4.1. *For any $\lambda \in \mathfrak{h}^*$, let*

$$W_{[\lambda]} = \{w \in W \mid w\lambda - \lambda \in R\} \qquad \Phi_{[\lambda]} = \{\beta \in \Phi \mid \langle \lambda, \beta^\vee \rangle \in \mathbb{Z}\}$$

where R is the root lattice.

Remark. Notice $\lambda \in \Lambda$ the weight lattice $\iff$ $W_{[\lambda]} = W$ and $\Phi_{[\lambda]} = \Phi$. Indeed if $\lambda \in \Lambda$, then

$$s_\alpha(\lambda) - \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha - \lambda = \langle \lambda, \alpha^\vee \rangle \alpha \in R \quad \forall \text{ simple } \alpha$$

and s_α generates W. To show $\Phi_{[\lambda]} = \Phi$ (aka $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all simple roots α actually implies $\langle \lambda, \beta^\vee \rangle$ for all roots β) requries a bit more work. Hint: Show $(s_\alpha(\beta))^\vee = s_\alpha(\beta^\vee)$ for any two roots α, β).

Thus, when $\lambda \notin \Lambda$ we see that $W_{[\lambda]}, \Phi_{[\lambda]}$ will be proper subsets of W, Φ.

Remark. Check that $w\rho - \rho \in R$ for any $w \in W$. Hint: w will send some positive roots to negative roots. But ρ is the half sum of <u>all</u> positive roots. In addition, check $\left\langle \rho, \alpha_i^\vee \right\rangle = 1$ for all simple roots α_i and thus $\rho \in \Lambda$. Hint: Compute $s_{\alpha_i}(\rho)$. As a result if follows that we can rephrase our definitions above as

$$W_{[\lambda]} = \{w \in W \mid w \star \lambda - \lambda \in R\} \qquad \Phi_{[\lambda]} = \left\{\beta \in \Phi \mid \left\langle \lambda + \rho, \beta^\vee \right\rangle \in \mathbb{Z}\right\}$$

Claim: A necessary condition for $L(\mu)$ to appear as a composition factor of $M(\lambda)$ is

$$\mu \leq \lambda \text{ and } \mu = w \star \lambda \text{ for some } \underline{w \in W_{[\lambda]}}$$

Proof. This is actually just combining our two previous conditions from before ($\mu \leq \lambda$ and $\mu = w \star \lambda$ for $w \in W$). Note $\mu \leq \lambda$ means $\mu - \lambda \in R^- \subset R$ and thus our combined necessary condition is that $w \star \lambda - \lambda \in R$. But this is exactly the statement that $w \in W_{[\lambda]}$ by the remark above.

Theorem 4.2. *Let $\lambda \in \mathfrak{h}^*$. Then*

(a) $\Phi_{[\lambda]}$ is a root system in it's $\mathbb{R}-$span.

(b) $W_{[\lambda]}$ is the Weyl group of the root system $\Phi_{[\lambda]}$. In particular it is generated by the reflections s_α where $\alpha \in \Phi_{[\lambda]}$.

Example 3. Consider $\mathfrak{g} = \mathfrak{sl}_3$ and let ω_1, ω_2 be the fundamental weights and let $\lambda = -0.5\omega_1 - 0.5\omega_2$.

[Draw picture]

Clearly $\lambda \notin \Lambda$. Note it's easier to compute $\Phi_{[\lambda]}$ first and then use the theorem above to compute $W_{[\lambda]}$. When testing for integrality ($\left\langle \lambda, \beta^\vee \right\rangle \in \mathbb{Z}$?) we can restrict our attention to the positive roots as the answer for the negative roots is the same as for the positive one. A_2 has 3 positive roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$. By definition we have that

$$\left\langle \lambda, \alpha_i^\vee \right\rangle = -0.5 \notin \mathbb{Z} \quad i = 1, 2$$

So we just need to compute

$$\left\langle -0.5\omega_1 - 0.5\omega_2, (\alpha_1 + \alpha_2)^\vee \right\rangle$$

Explicitly we have $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3$ so $\alpha_1 + \alpha_2 = \epsilon_1 - \epsilon_3$, and thus

$$(\alpha_1 + \alpha_2)^\vee = \frac{2(\epsilon_1 - \epsilon_3)}{\langle \epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_3 \rangle} = \frac{2(\epsilon_1 - \epsilon_3)}{2} = \epsilon_1 - \epsilon_3 = \alpha_1 + \alpha_2$$

And nearly the same computation shows that $\alpha_i^\vee = \alpha_i$ and thus we see that

$$\left\langle -0.5\omega_1 - 0.5\omega_2, (\alpha_1 + \alpha_2)^\vee \right\rangle = \left\langle -0.5\omega_1 - 0.5\omega_2, \alpha_1^\vee + \alpha_2^\vee \right\rangle = -1 \in \mathbb{Z}$$

Thus we see that $\Phi_{-0.5\omega_1 - 0.5\omega_2} = \{\alpha_1 + \alpha_2, -(\alpha_1 + \alpha_2)\}$ which is isomorphic to A_1 and thus $W_{-0.5\omega_1 - 0.5\omega_2} = \mathbb{Z}/2\mathbb{Z}$ (in fact is generated by $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$ in $W = S_3$). By our claim above, this means that $M(-0.5\omega_1 - 0.5\omega_2)$ can only have up to two different composition factors instead of up to 6.

5 $\rho-$Antidominant Weights

Definition 5.1. *A weight $\lambda \in \mathfrak{h}^*$ is called $\rho-$antidominant if*

$$\left\langle \lambda + \rho, \beta^\vee \right\rangle \notin \mathbb{Z}^{>0} \quad \underline{\forall \beta \in \Phi^+}$$

Similarly we say that a weight $\lambda \in \mathfrak{h}^*$ is $\rho-$dominant if

$$\left\langle \lambda + \rho, \beta^\vee \right\rangle \notin \mathbb{Z}^{<0} \quad \underline{\forall \beta \in \Phi^+}$$

Warning. $\rho-$antidominant is not the same as an antidominant weight in the usual sense nor is it the same as a $\rho-$shifted antidominant weight (aka $X^- - \rho$). [Draw picture] However $X^- - \rho \subset$ $\rho-$antidominant weights as given $\lambda \in X^-$ we have

$$\left\langle \lambda - \rho + \rho, \beta^\vee \right\rangle = \left\langle \lambda, \beta^\vee \right\rangle \leq 0$$

$\rho-$dominant is not the same as a dominant weight in the usual sense nor is it the same as a $\rho-$shifted dominant weight (aka $X^+ - \rho$). However $X^+ - \rho \subset \rho-$dominant weights.

Example 4. Let us find all $\rho-$antidominant weights of $\mathfrak{sl}_2$. Here there is only one positive root $\alpha = 2$ and thus $\alpha^\vee = 1$. Note that $\rho = 1$ also and thus

$$\lambda \text{ is } \rho-\text{antidominant} \iff \langle \lambda + 1, 1 \rangle = \lambda + 1 \notin \mathbb{Z}^{>0} \iff \lambda \neq 0, 1, 2, \ldots$$

And similarly, we have

$$\lambda \text{ is } \rho-\text{dominant} \iff \lambda \neq -2, -3, \ldots$$

Notice that these two sets are <u>not disjoint</u>, $-1 = -\rho$ is in both of them as well as all irrational numbers.

Example 5. Check that for $\mathfrak{g} = \mathfrak{sl}_3$ the weight $-0.5\omega_1 - 0.5\omega_2$ is not antidominant.

Proposition 5.2.

$$\lambda \text{ is } \rho-\text{antidominant} \iff \lambda \leq w \star \lambda \quad \forall w \in W_{[\lambda]}$$
$$\lambda \text{ is } \rho-\text{dominant} \iff \lambda \geq w \star \lambda \quad \forall w \in W_{[\lambda]}$$

Corollary 5
$\exists!$ $\rho-$antidominant weight in the orbit $W_{[\lambda]}\star\lambda$. Likewise $\exists!$ $\rho-$dominant weight in the orbit $W_{[\lambda]}\star\lambda$.

Proof. We just prove it for $\rho-$antidominant as $\rho-$dominant is very similar. We first show the existence of a $\rho-$antidominant weight in the orbit $W_{[\lambda]} \star \lambda$. Let μ be any weight in $W_{[\lambda]} \star \lambda$ that is minimal with respect to standard partial ordering. We claim μ is antidominant. Otherwise $\exists \beta \in \Phi^+$ s.t. $\left\langle \mu + \rho, \beta^\vee \right\rangle \in \mathbb{Z}^{>0}$. By our rephrased definitions we see that this means $\beta \in \Phi_{[\lambda]}$ and thus $s_\beta \in W_{[\lambda]}$. But now

$$s_\beta \star \mu - \mu = s_\beta(\mu + \rho) - \rho - \mu = \mu + \rho - \left\langle \mu + \rho, \beta^\vee \right\rangle \beta - \rho - \mu = -\left\langle \mu + \rho, \beta^\vee \right\rangle \beta \in R^-$$

But this exactly means that $s_\beta \star \mu < \mu$ contradicting our assumption that μ was minimal in $W_{[\lambda]} \star \lambda$.

Uniqueness follows immediately from the Proposition above.

6 Projectives in $\mathcal{O}$

Proposition 6.1. *(a) Suppose $\lambda \in \mathfrak{h}^*$ is $\rho-$dominant. Then $M(\lambda)$ is projective in $\mathcal{O}$.*

(b) If $P \in \mathcal{O}$ is projective and $\dim L < \infty$, then $P \otimes L$ is also projective in $\mathcal{O}$.

Proof. We want to construct a lift ψ of the following diagram of modules in $\mathcal{O}$

$$\begin{array}{ccccc} & & M(\lambda) & & \\ & \overset{\psi}{\swarrow} & \downarrow \varphi & & \\ M & \xrightarrow[\pi]{} & N & \longrightarrow & 0 \end{array}$$

(a) Let v_λ be the h.w. vector of $M(\lambda)$. Then $\varphi(v_\lambda)$ is a h.w. vector of weight λ in N. Again as $M \in \mathcal{O}$ and π is surjective, the same trick as in Proposition 1 shows that $v = \pi^{-1}(\varphi(v_\lambda)) \in M_\lambda$. Consider the submodule $U(\mathfrak{n}^+)v$. It is finite-dimensional as $M \in \mathcal{O}$. However since v is a weight vector, the action of all elements of $U(\mathfrak{n}^+)$ raises the weight and so to be f.d, we must have that there exists a h.w. vector say v_μ of weight $\mu \geq \lambda$ in $U(\mathfrak{n}^+)v$ and thus in M. This means we have a highest weight module S of weight μ occuring as a submodule of M that contains v. [Draw picture] We therefore have the following exact sequence

$$0 \to \ker \pi|_S \to S \xrightarrow{\pi|_S} \operatorname{im} \varphi \to 0$$

Therefore any composition factor of $\operatorname{im}\varphi$ appears as a composition factor of S. But $\operatorname{im}\varphi$ being the surjective image of a h.w. module of weight λ is also a h.w. module of weight λ and therefore $L(\lambda)$ is a composition factor of $\operatorname{im}\varphi$ and thus of S. But S is h.w. of weight μ and therefore a quotient of $M(\mu)$ and thus $L(\lambda)$ is a composition factor of $M(\mu)$ as well. But we know from before that a necessary condition is that $\lambda = w \star \mu \iff \mu = w^{-1} \star \lambda$ for some $w \in W_{[\lambda]}$. But λ is $\rho-$dominant and by Proposition 5.2 we see that this means $\mu \leq \lambda$ and thus $\mu = \lambda$. But this exactly means $\mathfrak{n}^+ v = 0$ or in other words v is a h.w. vector of weight λ in M and this is exactly what gives us the lift $\psi : M(\lambda) \to M$ above.

(b) Left to reader, note that since $\dim L < \infty$, we know $P \otimes L$ is in $\mathcal{O}$. Hint: Use Tensor-Hom and note that the inclusion functor $\iota : \mathcal{O} \hookrightarrow U(\mathfrak{g}) - \text{mod}$ is fully faithful.

Remark. Notice that as a special case of part (a) above we can re derive Proposition 1 part (a), aka $\operatorname{Ext}^1(M(\lambda), M) = 0$ if M is h.w. of weight μ where $\mu \leq \lambda$. This also means that for general λ, $M(\lambda)$ is projective in the Serre subcategory generated by $L(\mu)$ where $\mu \leq \lambda$.

Remark. $M(\lambda)$ is not projective as a $U(\mathfrak{g})$ module, indeed our example above with $M \notin \mathcal{O}$ gives us a SES of $U(\mathfrak{sl}_2)$ modules that is not split with $M(\lambda)$ occuring as the quotient and thus can't be projective.

Remark. As a clarification, note that

$$\text{Any block of Category } \mathcal{O} \subseteq \mathcal{O}_{\chi_\lambda}$$

as $\operatorname{Ext}^i_{\mathcal{O}}(A, B) = 0$ for $A \in \mathcal{O}_{\chi_\lambda}, B \in \mathcal{O}_{\chi_\mu}$ because the central character acts differently on A, B. We now give an example where this is a proper inclusion, aka $\mathcal{O}_{\chi_\lambda}$ will have more than one block in it.

Let $\mathfrak{g} = \mathfrak{sl}_2$ and let $\lambda \notin \mathbb{Q}$. Check that for $\mathfrak{g} = \mathfrak{sl}_2$, that $M(\lambda)$ is simple $\iff \lambda \notin \mathbb{Z}^{\geq 0}$. Therefore $M(\lambda), M(s \star \lambda) = M(-\lambda-2)$ are both simple and thus $\operatorname{Hom}_{\mathcal{O}}(M(\lambda), M(-\lambda-2)) = 0$. But from above we know that λ is also $\rho-$dominant and thus $M(\lambda)$ is projective and therefore

$$\operatorname{Ext}^i_{\mathcal{O}}(M(\lambda), M(-\lambda-2)) = 0 \quad \forall i \geq 1$$

So even though $M(\lambda)$ and $M(-\lambda-2)$ are in the same linkage class, they are not in the same block!